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On the Rate of Convergence of the Generalized Durrmeyer Type Operators for Functions of Bounded Variation[1]

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In this paper we estimate the rate of convergence of the Durrmeyer–Bézier operators $D_{n,\alpha}(f, x)$ for functions of bounded variation and prove that the $D_{n,\alpha}(f, x)$ converge to the limit $1/(\alpha+1) f(x+) + \alpha/(\alpha+1) f(x-)$ for functions of bounded variation $f(t)$. Our result improves and extends the result of S. Guo (1987, *J. Approx. Theory* **51**, 183–192). © 2000 Academic Press

1. INTRODUCTION

For a function f defined on $[0, 1]$, the Bernstein operator B_n is defined by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, p_{nk}(x), \qquad p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

A modified type of operator B_n is defined by

$$B_{n,\alpha}(f, x) = \sum_{k=0}^{n} f(k/n)\, Q_{nk}^{(\alpha)}(x), \tag{2}$$

where $\alpha \geqslant 1$ and $Q_{nk}^{(\alpha)}(x) = (\sum_{j=k}^{n} p_{nj}(x))^{\alpha} - (\sum_{j=k+1}^{n} p_{nj}(x))^{\alpha}$.

The operator $B_{n,\alpha}(f)$ is called the Bernstein–Bézier operator. When $\alpha = 1$, $B_{n,1}(f)$ is just the operator $B_n(f)$ given as (1).

Naturally, the same modification can be done for Szász, Kantorovich, and Durrmeyer operators. These modified types of operators are called the Szász–Bézier operator $S_{n,\alpha}(f, x)$, Kantorovich–Bézier operator $L_{n,\alpha}(f, x)$,

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and Durrmeyer–Bézier operator $D_{n,\alpha}(f, x)$, respectively. In the concrete, for a function f defined on $[0, 1]$ and $\alpha \geqslant 1$, the Durrmeyer–Bézier operator $D_{n,\alpha}$ is defined by

$$D_{n,\alpha}(f, x) = (n+1) \sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x) \int_0^1 f(t)\, p_{nk}(t)\, dt, \tag{3}$$

where $p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $Q_{nk}^{(\alpha)}(x) = (\sum_{j=k}^{n} p_{nj}(x))^\alpha - (\sum_{j=k+1}^{n} p_{nj}(x))^\alpha$.

It is obvious that $D_{n,\alpha}$ is a positive linear operator and $D_{n,\alpha}(1, x) = 1$. When $\alpha = 1$, $D_{n,1}(f)$ is just the well-known Durrmeyer operator

$$D_{n,1}(f, x) = (n+1) \sum_{k=0}^{n} p_{nk}(x) \int_0^1 f(t)\, p_{nk}(t)\, dt. \tag{4}$$

Cheng [4,5] and Guo [6] estimated the rates of convergence of Bernstein operators $B_n(f, x)$, Szász operators $S_n(f, x)$ and Durrmeyer operators $D_n(f, x)$ for functions of bounded variation and proved that the operators $B_n(f, x)$, $S_n(f, x)$ and $D_n(f, x)$ all convergence to the limit $\frac{1}{2} f(x+) + \frac{1}{2} f(x-)$ for functions of bounded variation. References [2, 3] proved that the operators $B_{n,\alpha}(f, x)$, $L_{n,\alpha}(f, x)$ and $S_{n,\alpha}(f, x)$ all convergence to the limit $(1/2^\alpha) f(x+) + (1 - 1/2^\alpha) f(x-)$ for functions of bounded variation ($\alpha \geqslant 1, n \to +\infty$). In this paper we shall estimate the rate of convergence of operators $D_{n,\alpha}(f, x)$ for functions of bounded variation and prove that operators $D_{n,\alpha}(f, x)$ converge to the limit $(1/(\alpha+1)) f(x+) + (\alpha/(\alpha+1)) f(x-)$ for functions of bounded variation. This result is unforeseen, which shows that the approximation property of $D_{n,\alpha}(f, x)$ is different from those of $B_{n,\alpha}(f, x)$, $L_{n,\alpha}(f, x)$ and $S_{n,\alpha}(f, x)$ in essence. Our main result can be stated as follows:

THEOREM. *Let f be a function of bounded variation on $[0, 1]$ ($f \in BV[0, 1]$), $\alpha \geqslant 1$. Then for every $x \in (0, 1)$ and $n > (1/x(1-x))$ we have*

$$\left| D_{n,\alpha}(f, x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right|$$
$$\leqslant \frac{8\alpha}{nx(1-x)} \sum_{k=1}^{n} \bigvee_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}} (g_x) + \frac{2\alpha}{\sqrt{nx(1-x)}} |f(x+) - f(x-)|, \tag{5}$$

where $\bigvee_a^b (g_x)$ is the total variation of g_x on $[a, b]$ and

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leqslant 1; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leqslant t < x. \end{cases} \tag{6}$$

2. PRELIMINARY RESULTS

We need some preliminary results for proving Theorem. First, with the method of Bojanic and Vuilleumier [1] (see Cheng [4] and Guo [6]), we prove the following:

LEMMA 1. *For every* $x \in (0, 1)$ *and* $n > 1/x(1-x)$, *there holds*

$$|D_{n,\alpha}(g_x, x)| \leqslant \frac{8\alpha}{nx(1-x)} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x). \tag{7}$$

Proof. Let $K_{n,\alpha}(x, t) = (n+1) \sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x)\, p_{nk}(t)$. We first recall the Lebesgue–Stieltjes integral representations:

$$D_{n,\alpha}(g_x, x) = \int_0^1 g_x(t)\, K_{n,\alpha}(x, t)\, dt. \tag{8}$$

We decompose the integral of (8) into three parts, as

$$\int_0^1 g_x(t)\, K_{n,\alpha}(x, t)\, dt = \triangle_{1,n}(f, x) + \triangle_{2,n}(f, x) + \triangle_{3,n}(f, x),$$

where $\triangle_{1,n}(f, x) = \int_0^{x-x/\sqrt{n}} g_x(t)\, K_{n,\alpha}(x, t)\, dt$, $\triangle_{2,n}(f, x) = \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(t)$ $K_{n,\alpha}(x, t)\, dt$, $\triangle_{3,n}(f, x) = \int_{x+(1-x)/\sqrt{n}}^{1} g_x(t)\, K_{n,\alpha}(x, t)\, dt$. Let $\lambda_{n,\alpha}(x, t) = \int_0^t K_{n,\alpha}(x, u)\, du$. First, we estimate $\triangle_{2,n}(f, x)$. For $t \in [x - x/\sqrt{n}, x + (1-x)/\sqrt{n}]$, we have

$$\begin{aligned} |\triangle_{2,n}(f, x)| &= \left| \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x(t) - g_x(x))\, d_t \lambda_{n,\alpha}(x, t) \right| \\ &\leqslant \bigvee_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x) \leqslant \frac{1}{n-1} \sum_{k=2}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x). \end{aligned} \tag{9}$$

Next, we estimate $\triangle_{1,n}(f, x)$. Let $y = x - x/\sqrt{n}$. Using partial Lebesgue–Stieltjes integration, we have

$$\begin{aligned} \triangle_{1,n}(f, x) &= \int_0^{x-x/\sqrt{n}} g_x(t)\, d_t \lambda_{n,\alpha}(x, t) \\ &= g_x(y+)\, \lambda_{n,\alpha}(x, y) - \int_0^y \lambda_{n,\alpha}(x, t)\, d_t\, g_x(t). \end{aligned}$$

Since $|g_x(y+)|=|g_x(y+)-g_x(x)|\leqslant\bigvee_{y+}^{x}(g_x)$, it follows that

$$|\triangle_{1,n}(f,x)|\leqslant\bigvee_{y+}^{x}(g_x)\,\lambda_{n,\alpha}(x,y)+\int_0^y\lambda_{n,\alpha}(x,t)\,d_t\left(-\bigvee_t^x(g_x)\right). \tag{10}$$

Using the fact that $|a^\alpha-b^\alpha|\leqslant\alpha\,|a-b|$ with $0\leqslant a,\ b\leqslant 1$, and $\alpha\geqslant 1$, we get $Q_{nk}^{(\alpha)}(x)\leqslant\alpha p_{nk}(x)$. By this inequality and the proof of Lemma II.1. of [10, p. 327] (cf. [6, p. 186] also), it follows that

$$\begin{aligned}D_{n,\alpha}((t-x)^2,x)&\leqslant\alpha D_{n,1}((t-x)^2,x)\\&=\alpha\frac{2(n-3)\,x(1-x)+2}{(n+2)(n+3)}\leqslant\alpha\frac{2nx(1-x)+2}{n^2}.\end{aligned}$$

Note that $y=x-x/\sqrt{n}\leqslant x$, hence

$$\begin{aligned}\lambda_{n,\alpha}(x,y)&=\int_0^y K_{n,\alpha}(x,t)\,dt\leqslant\int_0^y\left(\frac{x-t}{x-y}\right)^2K_{n,\alpha}(x,t)\,dt\\&\leqslant\frac{1}{(x-y)^2}D_{n,\alpha}((t-x)^2,x)\leqslant\alpha\frac{2nx(1-x)+2}{n^2(x-y)^2}.\end{aligned} \tag{11}$$

From (10) and (11) we have

$$\begin{aligned}|\triangle_{1,n}(f,x)|&\leqslant\alpha\frac{2nx(1-x)+2}{n^2(x-y)^2}\bigvee_{y+}^{x}(g_x)\\&\quad+\alpha\frac{2nx(1-x)+2}{n^2}\int_0^y\frac{1}{(x-t)^2}\,d_t\left(-\bigvee_t^x(g_x)\right).\end{aligned} \tag{12}$$

Furthermore, since

$$\int_0^y\frac{1}{(x-t)^2}\,d_t\left(-\bigvee_t^x(g_x)\right)=-\frac{1}{(x-t)^2}\bigvee_t^x(g_x)\Big|_0^{y+}+\int_0^y\frac{2}{(x-t)^3}\bigvee_t^x(g_x)\,dt.$$

So we have from (12)

$$|\triangle_{1,n}(f,x)|\leqslant\alpha\frac{2nx(1-x)+2}{n^2}\left(\bigvee_0^x(g_x)/x^2+\int_0^{x-x/\sqrt{n}}\frac{2}{(x-t)^3}\bigvee_t^x(g_x)\,dt\right).$$

Putting $t=x-x/\sqrt{u}$ in the last integral, we get

$$\int_0^{x-x/\sqrt{n}}\frac{2}{(x-t)^3}\bigvee_t^x(g_x)\,dt=\frac{1}{x^2}\int_1^n\bigvee_{x-x/\sqrt{u}}^{x}(g_x)\,du\leqslant\frac{1}{x^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x).$$

Consequently

$$
\begin{aligned}
|\triangle_{1,n}(f,x)| &\leqslant \alpha \frac{2nx(1-x)+2}{n^2x^2}\left(\bigvee_0^x (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x} (g_x)\right) \\
&\leqslant \alpha \frac{2nx(1-x)+2}{n^2x^2}\left(\bigvee_0^1 (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x)\right). \qquad (13)
\end{aligned}
$$

Using the similar method for estimating $|\triangle_{3,n}(f,x)|$, we get

$$
\begin{aligned}
|\triangle_{3,n}(f,x)| &\leqslant \alpha \frac{2nx(1-x)+2}{n^2(1-x)^2}\left(\bigvee_x^1 (g_x) + \sum_{k=1}^n \bigvee_{x}^{x+(1-x)/\sqrt{k}} (g_x)\right) \\
&\leqslant \alpha \frac{2nx(1-x)+2}{n^2(1-x)^2}\left(\bigvee_0^1 (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x)\right). \qquad (14)
\end{aligned}
$$

Note that $(1-x)^2+x^2 \leqslant 1$, and it is easy to verify that $1/(n-1) \leqslant \alpha((2nx(1-x)+2)/n^2x^2(1-x)^2)$, for $n>1$, $x \in [0,1]$. Hence from (9), (13), and (14), it follows that

$$
\begin{aligned}
|D_{n,\alpha}(g_x,x)| &\leqslant |\triangle_{1,n}(f,x)| + |\triangle_{2,n}(f,x)| + |\triangle_{3,n}(f,x)| \\
&\leqslant \frac{1}{n-1}\sum_{k=2}^n \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x) + \alpha \frac{2nx(1-x)+2}{n^2x^2(1-x)^2} \\
&\quad \times \left(\bigvee_0^1 (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x)\right) \\
&\leqslant \alpha \frac{4nx(1-x)+4}{n^2x^2(1-x)^2} \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x). \qquad (15)
\end{aligned}
$$

Obviously, $\alpha((4nx(1-x)+4)/n^2x^2(1-x)^2) < 8\alpha/nx(1-x)$, for $n>1/x(1-x)$. Hence Lemma 1 follows from (15).

Lemma 2 is the well-known Berry–Esseen bound for the classical central limit theorem of probability theory. Its proof and further discussion can be found in Shiryayev [7, p. 432].

LEMMA 2. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of independent and identically distributed random variables with finite variance such that the expectation* $E(\xi_1) = a_1 \in R = (-\infty, +\infty)$, *the variance* $Var(\xi_1) = b_1^2 > 0$. *Assume* $E|\xi_1 - a_1|^3 < \infty$. *Then there exists a constant* C, $1/\sqrt{2\pi} \leqslant C < 0.8$, *such that for all* $n = 1, 2, 3, \cdots$ *and all* t

$$
\left| P\left(\frac{1}{b_1\sqrt{n}}\sum_{k=1}^n (\xi_k - a_1) \leqslant t\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-u^2/2}\,du \right| < C\frac{E|\xi_1-a_1|^3}{\sqrt{n}b_1^3}. \qquad (16)
$$

LEMMA 3. *For* $j=1, 2, 3, ..., n$, *and* $x\in(0,1)$, *writing* $A_1=(j-1-nx)/\sqrt{nx(1-x)}$, $A_2=(j-(n+1)\,x)/\sqrt{(n+1)\,x(1-x)}$, *and* $A_3=(j-1-(n+1)\,x)/\sqrt{(n+1)\,x(1-x)}$, *we have*

$$|A_1-A_2|\leqslant\frac{1}{\sqrt{nx(1-x)}}, \qquad \text{and} \qquad |A_1-A_3|\leqslant\frac{1}{\sqrt{nx(1-x)}}. \tag{17}$$

Proof. It is evident for $j=1, 2, 3, ..., n$, and $x\in(0,1)$ that

$$0\leqslant\frac{j}{\sqrt{n+1}}+\sqrt{n}x\leqslant\frac{n}{\sqrt{n+1}}+\sqrt{n}\leqslant 2(\sqrt{n+1}+\sqrt{n}),$$

which implies

$$0\leqslant\frac{j}{\sqrt{n+1}}(\sqrt{n+1}-\sqrt{n})+\sqrt{n}x(\sqrt{n+1}-\sqrt{n})\leqslant 2.$$

Hence

$$\left|\frac{j}{\sqrt{n+1}}(\sqrt{n+1}-\sqrt{n})+\sqrt{n}x(\sqrt{n+1}-\sqrt{n})-1\right|\leqslant 1.$$

That is,

$$\left|j-1-nx-\frac{\sqrt{n}}{\sqrt{n+1}}(j-(n+1)\,x)\right|\leqslant 1.$$

Consequently

$$|A_1-A_2|\leqslant\frac{1}{\sqrt{nx(1-x)}}.$$

The second inequality in (17) can be obtained in the same manner.

Set

$$J_{nk}(x)=\sum_{j=k}^{n} p_{nj}(x) \qquad (J_{n,\,n+1}(x)\equiv 0). \tag{18}$$

Then for $k=0, 1, 2, 3, ..., n$

$$J'_{n+1,k+1}(x)=\sum_{j=k+1}^{n+1} p'_{n+1,j}(x)$$
$$=(n+1)\sum_{j=k+1}^{n}(p_{n,j-1}(x)-p_{nj}(x))+(n+1)\,p_{nn}(x)$$
$$=(n+1)\,p_{nk}(x)$$

and $J_{n+1,k+1}(0)=0$. So we have

$$(n+1)\int_0^x p_{nk}(t)\,dt=J_{n+1,k+1}(x)=1-\sum_{j=0}^{k} p_{n+1,j}(x). \tag{19}$$

By Lemmas 2, 3, we prove the

LEMMA 4. *For all $x\in(0,1)$ and $j=0, 1, 2, 3, ..., n$, we have*

$$|J^{\alpha}_{nj}(x)-J^{\alpha}_{n+1,j+1}(x)|\leqslant\frac{2\alpha}{\sqrt{nx(1-x)}}, \tag{20}$$

and

$$|J^{\alpha}_{nj}(x)-J^{\alpha}_{n+1,j}(x)|\leqslant\frac{2\alpha}{\sqrt{nx(1-x)}} \tag{21}$$

Proof. First, $|J^{\alpha}_{n0}(x)-J^{\alpha}_{n+1,0}(x)|=1-1=0$, so (21) holds for $j=0$, and $|J^{\alpha}_{n0}(x)-J^{\alpha}_{n+1,1}(x)|\leqslant\alpha\,|J_{n0}(x)-J_{n+1,1}(x)|=\alpha p_{n+1,0}(x)$, so (20) holds for $j=0$ from the fact that $p_{n+1,0}(x)\leqslant 2/\sqrt{nx(1-x)}$ (cf. [11, Theorem 1]). For $j=1, 2, 3, ..., n$, let ξ_1 be the random variable with two-point distribution $P(\xi_1=i)=x^i(1-x)^{1-i}$ ($i=0, 1$, and $x\in[0,1]$ is a parameter). Hence

$$a_1=E(\xi_1)=x, \qquad b_1^2=E(\xi_1-a_1)^2=x(1-x), \tag{22}$$

and $E(\xi_1-a_1)^4=x(1-x)-3x^2(1-x)^2$ (cf. [8, p. 14]). By Hölder inequality, we get

$$E\,|\xi_1-a_1|^3\leqslant\sqrt{E(\xi_1-a_1)^4\,E(\xi_1-a_1)^2}=\sqrt{x^2(1-x)^2-3x^3(1-x)^3}$$
$$=x(1-x)\sqrt{1-3x(1-x)}\leqslant x(1-x). \tag{23}$$

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent random variables identically distributed with ξ_1, $\eta_n = \sum_{k=1}^{n} \xi_k$. Then the probability distribution of the random variable η_n is

$$P(\eta_n = k) = \binom{n}{k} x^k (1-x)^{n-k} = p_{nk}(x) \qquad (0 \leqslant k \leqslant n).$$

So

$$\begin{aligned} |J_{nj}(x) - J_{n+1,\, j+1}(x)| &= |P(j \leqslant \eta_n \leqslant n) - P(j+1 \leqslant \eta_{n+1} \leqslant n+1)| \\ &= |1 - P(\eta_n \leqslant j-1) - 1 + P(\eta_{n+1} \leqslant j)| \\ &= |P(\eta_n \leqslant j-1) - P(\eta_{n+1} \leqslant j)|. \end{aligned} \tag{24}$$

Writing $A_1 = (j-1-nx)/\sqrt{nx(1-x)}$, $A_2 = (j-(n+1)\, x)/\sqrt{(n+1)\, x(1-x)}$ and using Lemma 2 and (22), (23), we have

$$\left| P(\eta_n \leqslant j-1) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1} e^{-t^2/2}\, dt \right| \leqslant C \frac{E\, |\xi_1 - a_1|^3}{\sqrt{n b_1^3}} \leqslant \frac{0.8}{\sqrt{nx(1-x)}} \tag{25}$$

and

$$\left| P(\eta_{n+1} \leqslant j) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_2} e^{-t^2/2}\, dt \right| \leqslant \frac{0.8}{\sqrt{nx(1-x)}}. \tag{26}$$

From (24)–(26) and Lemma 3 we get

$$\begin{aligned} |J_{nj}(x) - J_{n+1,\, j+1}(x)| &\leqslant \left| \frac{1}{\sqrt{2\pi}} \int_{A_1}^{A_2} e^{-t^2/2}\, dt \right| + \frac{1.6}{\sqrt{nx(1-x)}} \\ &\leqslant \frac{1}{\sqrt{2\pi}} |A_1 - A_2| + \frac{1.6}{\sqrt{nx(1-x)}} \\ &\leqslant \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{nx(1-x)}} + \frac{1.6}{\sqrt{nx(1-x)}} \\ &< \frac{2}{\sqrt{nx(1-x)}}. \end{aligned}$$

Hence from the inequality $|a^\alpha - b^\alpha| \leqslant \alpha\, |a-b|$ with $0 \leqslant a, b \leqslant 1$, and $\alpha \geqslant 1$, it follows that

$$|J_{nj}^{\alpha}(x) - J_{n+1,\, j+1}^{\alpha}(x)| \leqslant \frac{2\alpha}{\sqrt{nx(1-x)}}.$$

The proof of (21) is similar. ∎

3. PROOF OF THE THEOREM AND REMARKS

For any $f(t) \in BV[0,1]$, it is known that

$$f(t) = \frac{1}{2} f(x+) + \frac{1}{2} f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2} \operatorname{sgn}(t-x) + \delta_x(t) \times \left[f(x) - \frac{1}{2} f(x+) - \frac{1}{2} f(x-) \right], \tag{27}$$

where $g_x(t)$ is defined as (6) and

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0; \end{cases} \qquad \delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x. \end{cases}$$

By a simple algebra (27) can be expressed as

$$f(t) = \frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) + g_x(t) + \frac{f(x+) - f(x-)}{2} \times \left[\operatorname{sgn}(t-x) + \frac{\alpha-1}{\alpha+1} \right] + \delta_x(t) \left[f(x) - \frac{1}{2} f(x+) - \frac{1}{2} f(x-) \right]. \tag{28}$$

Obviously, $D_{n,\alpha}(\delta_x, x) = 0$, hence we have

$$\left| D_{n,\alpha}(f,x) - \left[\frac{1}{\alpha+1} f(x+) + \frac{\alpha}{\alpha+1} f(x-) \right] \right| \leqslant |D_{n,\alpha}(g_x, x)| + \frac{|f(x+) - f(x-)|}{2} \left| D_{n,\alpha}(\operatorname{sgn}(t-x), x) + \frac{\alpha-1}{\alpha+1} \right|.$$

From Lemma 1, it is clear that our theorem will be proved if we establish that

$$\left| D_{n,\alpha}(\operatorname{sgn}(t-x), x) + \frac{\alpha-1}{\alpha+1} \right| \leqslant \frac{2(2\alpha)}{\sqrt{nx(1-x)}}.$$

In fact, by direct calculation and using (19) we have

$$D_{n,\alpha}(\operatorname{sgn}(t-x),x)=(n+1)\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\left(\int_x^1 p_{nk}(t)\,dt-\int_0^x p_{nk}(t)\,dt\right)$$
$$=(n+1)\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\left(\int_0^1 p_{nk}(t)\,dt-2\int_0^x p_{nk}(t)\,dt\right)$$
$$=1-2(n+1)\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\int_0^x p_{nk}(t)\,dt$$
$$=1-2\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\left(1-\sum_{j=0}^{k}p_{n+1,j}(x)\right)$$
$$=-1+2\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\sum_{j=0}^{k}p_{n+1,j}(x).$$

Noticing that $\sum_{k=0}^{n}\sum_{j=0}^{k}\bullet=\sum_{j=0}^{n}\sum_{k=j}^{n}\bullet$, $\sum_{k=j}^{n}Q_{nk}^{(\alpha)}(x)=J_{nj}^{\alpha}(x)$ and $J_{n,n+1}(x)\equiv 0$, we have

$$-1+2\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\sum_{j=0}^{k}p_{n+1,j}(x)$$
$$=-1+2\sum_{j=0}^{n}p_{n+1,j}(x)\sum_{k=j}^{n}Q_{nk}^{(\alpha)}(x)$$
$$=-1+2\sum_{j=0}^{n}p_{n+1,j}(x)\,J_{nj}^{\alpha}(x)$$
$$=-1+2\sum_{j=0}^{n+1}p_{n+1,j}(x)\,J_{nj}^{\alpha}(x).$$

Hence

$$D_{n,\alpha}(\operatorname{sgn}(t-x),x)+\frac{\alpha-1}{\alpha+1}=2\sum_{j=0}^{n+1}p_{n+1,j}(x)\,J_{nj}^{\alpha}(x)-\frac{2}{\alpha+1}$$
$$=2\sum_{j=0}^{n+1}p_{n+1,j}(x)\,J_{nj}^{\alpha}(x)-\frac{2}{\alpha+1}\sum_{j=0}^{n+1}Q_{n+1,j}^{(\alpha+1)}(x). \tag{29}$$

By the mean value theorem

$$Q_{n+1,j}^{(\alpha+1)}(x)=J_{n+1,j}^{\alpha+1}(x)-J_{n+1,j+1}^{\alpha+1}(x)=(\alpha+1)\,p_{n+1,j}(x)\,\gamma_{nj}^{\alpha}(x),$$

where $J_{n+1,j+1}(x)<\gamma_{nj}(x)<J_{n+1,j}(x)$. Hence it follows from Lemma 4 and (29) that

$$\left|D_{n,\alpha}(\operatorname{sgn}(t-x),x)+\frac{\alpha-1}{\alpha+1}\right|=\left|2\sum_{j=0}^{n+1}p_{n+1,j}(x)(J_{nj}^{\alpha}(x)-\gamma_{nj}^{\alpha}(x))\right|$$
$$\leqslant 2\sum_{j=0}^{n+1}p_{n+1,j}(x)\frac{2\alpha}{\sqrt{nx(1-x)}}$$
$$=\frac{2(2\alpha)}{\sqrt{nx(1-x)}}.$$

The proof is complete.

Remark 1. We shall show that our estimate is the best possible for continuity points of bounded variation function *f*. If x is a continuity point of *f*, (5) becomes

$$|D_{n,\alpha}(f,x)-f(x)|\leqslant\frac{8\alpha}{nx(1-x)}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(f). \tag{30}$$

When $\alpha=1$, the conclusion is known (see [6, Remark]). For $\alpha\neq 1$, consider the function $f(t)=t$. From (30) we have

$$|D_{n,\alpha}(t,x)-x|\leqslant\frac{8\alpha}{nx(1-x)}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\leqslant\frac{16\alpha}{\sqrt{nx(1-x)}}. \tag{31}$$

On the other hand, From [2, (38)], there exists a positive constant C_1 such that

$$\left|\sum_{k=0}^{n}\frac{k}{n}Q_{nk}^{(\alpha)}(x)-x\right|\geqslant C_1\frac{\sqrt{x(1-x)}}{\sqrt{n}},\qquad\text{for } n \text{ sufficiently large.} \tag{32}$$

Hence

$$|D_{n,\alpha}(t,x)-x|=\left|(n+1)\sum_{k=0}^{n}Q_{nk}^{(\alpha)}(x)\int_0^1 tp_{nk}(t)\,dt-x\right|$$
$$=\left|\sum_{k=0}^{n}\frac{k+1}{n+2}Q_{nk}^{(\alpha)}(x)-x\right|$$
$$=\left|\frac{n}{n+2}\left[\sum_{k=0}^{n}\frac{k}{n}Q_{nk}^{(\alpha)}(x)-x\right]+\frac{1-2x}{n+2}\right|$$
$$\geqslant\left|\frac{n}{n+2}\left[\sum_{k=0}^{n}\frac{k}{n}Q_{nk}^{(\alpha)}(x)-x\right]\right|-\frac{1}{n+2}$$
$$\geqslant C_1\frac{\sqrt{x(1-x)}}{2\sqrt{n}}, \tag{33}$$

for n sufficiently large such that (32) holds and meanwhile such that $1/n + 1/C_1\sqrt{x(1-x)}\sqrt{n} \leqslant 1/2$. Hence by (31) and (33), we know that (30) cannot be asymptotically improved.

Remark 2. We conjecture that the second term on the right-hand side of (5) is asymptotically optimal also ($n \to +\infty$). Similar results have been obtained for operators $B_{n,\alpha}$, $L_{n,\alpha}$ and $S_{n,\alpha}$ (see [2, 3]).

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